Question 1

> Prove that $\sqrt[3]{361}$ is irrational.

To determine whether the cube root of 361 is irrational, we need to examine whether it can be expressed as a rational number, which is a number that can be written as the quotient of two integers (where the denominator is not zero).
Let us assume that $\sqrt[3]{361}$ is a rational number. As such, $\sqrt[3]{361} = \frac{p}{q}$, where p,q are co-prime integers, and $q \neq 0$ (definition of a rational number).

Taking the cube of both sides of this equation gives

$$\begin{aligned} 361 &= \left(\frac{p}{q}\right)^3 \\ &= \frac{p^3}{q^3} \\ p^3 &= 361q^3 \quad (1) \end{aligned}$$

Since p and q are integers, p^3 and q^3 are also integers.

Consider the prime factorisation $361 = 19 \times 19$. 19 is prime, appears twice in the factorisation, and also is the only number in the factorisation. It is not in the form $\frac{p^3}{q^3}$. Therefore, there is no way to express 361 as the cube of a rational number.

This contradicts our original assumption that $\sqrt[3]{361}$ is rational.

Hence, by contradiction, we can conclude that $\sqrt[3]{361}$ is irrational.

Question 2

> A relation $\star$ is defined on the set $\mathbb{Q}^2$ by
> $(x_1, x_2) \star (y_1, y_2)$ if and only if $x_1y_2 = x_2y_1$.
>
> For each of the questions below, be sure to provide a proof supporting your answer.

Part A

> Is $\star$ reflexive?

A relation on $\mathbb{Q}^2$ is reflexive if $a \star a$ for all $a \in \mathbb{Q}^2$. To prove reflexivity, assume that $x_1 x_2 = y_2 y_1$ is true (let this be (1)).

Given that $x_1 y_2 = x_2 y_1$ is known to be true, we can rearrange for y_1 to find that $y_1 = \frac{x_1 y_2}{x_2}(2)$. In a similar manner, rearranging for y_2 provides $y_2 = \frac{x_2 y_1}{x_1}(3)$. Substituting (2) and (3) into (1), it can be found that $x_2 = y_2$ and $x_1 = y_1$. Therefore, $x_1 x_2 = y_2 y_1 = x_1 y_2 = x_2 y_1$. Since every (x_1, x_2) satisfies the requirement that $x_1 x_2 = y_2 y_1$, the relation is reflexive.

Part B

> Is $\star$ symmetric?

For $\star$ to be symmetric, every set of elements $(x_1, x_2) \in \mathbb{Q}^2, (y_1, y_2) \in \mathbb{Q}^2$ which satisfies $(x_1, x_2) \star (y_1, y_2)$ must also satisfy $(y_1, y_2) \star (x_1, x_2)$.

Assume $(y_1, y_2) \star (x_1, x_2)$ is true. This means that $y_1 x_2 = y_2 x_1$, which is the same as as $x_1 y_2 = x_2 y_1$ (i.e. the original relation). Therefore the requirement is satisfied, and $\star$ is symmetric.

Part C

> Is $\star$ anti-symmetric?

If $\star$ is anti-symmetric, then for every distinct pair of (x_1, x_2) and (y_1, y_2) in $\mathbb{Q}^2$ such that $(x_1, x_2) \star (y_1, y_2)$ and $(y_1, y_2) \star (x_1, x_2)$, $x_1 = y_1$ and $x_2 = y_2$.

Given that $x_1 y_2 = x_2 y_1$ is known to be true, we can rearrange for y_1 to find that $y_1 = \frac{x_1 y_2}{x_2}(2)$. In a similar manner, rearranging for y_2 provides $y_2 = \frac{x_2 y_1}{x_1}(3)$. Substituting (2) and (3) into (1), it can be found that $x_2 = y_2$ and $x_1 = y_1$. Therefore, the relation is anti-symmetric.

Part D

> Is $\star$ transitive?

For $\star$ to be transitive, then for any set of 3 elements (x_1, x_2), (y_1, y_2), and (z_1, z_2) where $(x_1, x_2) \star (y_1, y_2)$ and $(y_1, y_2) \star (z_1, z_2)$ are both true, then $(x_1, x_2) \star (z_1, z_2)$ must also be true.

Consider a set of 3 elements (x_1, x_2), (y_1, y_2), and (z_1, z_2) where $(x_1, x_2) \star (y_1, y_2)$ and $(y_1, y_2) \star (z_1, z_2)$ are both true. This means that $x_1 y_2 = x_2 y_1$ and $y_1 z_2 = y_2 z_1$. From this, it can be found that $y_1 = \frac{y_2 z_1}{z_2} = \frac{x_1 y_2}{x_2}$. This can be simplified to

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{x_1}{x_2} \\ x_1 &= \frac{z_1 x_2}{z_2} \\ x_1 z_2 &= z_1 x_2, \end{aligned}$$

which satisfies the requirement for transitivity. Therefore, the relation $\star$ is transitive.

Part E

> Is $\star$ an equivalence relation, a partial order, both, or neither?

A relation is an equivalence relation if it is reflexive, symmetric and transitive. It was shown in parts A, B, and D that $\star$ meets these criteria.

A relation is a partial order if it is reflexive, anti-symmetric, and transitive. It was shown in parts A, B, and C that $\star$ meets these criteria.

Therefore, $\star$ is both an equivalence relation and a partial order.

Question 3

> Show that for all integers n:

Part A

> If d is an integer such that $d|n+5$ and $d|n^2+2$, then $d|27$.

Let d be an integer. The three requirements for d are that

$$\begin{aligned} d|n+5 &\quad (1) \\ d|n^2+2 &\quad (2) \\ d|27 &\quad (3), \end{aligned}$$

Where n is any integer.

Assume that (1) and (2) are true. The consequence of this is that there must be 2 integers, (let them be a and b), such that

$$\begin{aligned} n+5 &= ad \quad (3), \text{derived from } (1) \\ n^2+2 &= bd \quad (4), \text{derived from } (2). \end{aligned}$$

Equation (3) can be rearranged such that $n = ad - 5$. Substituting this into (4),

$$\begin{aligned} bd &= (ad+5)^2 + 2 \\ &= a^2d^2 - 10ad + 25 + 2 \\ 27 &= bd - a^2d^2 + 10ad \\ 27 &= d\left(b - a^2d + 10a\right). \qquad (5) \end{aligned}$$

Since a,b, and d are all defined as integers, the right hand side of (5) is an integer which is divisible by d. Therefore, 27 must also be divisible by d, satisfying (3). As such, we can conclude that for all integers n, if d is an integer such that $d|n+5$ and $d|n^2+2$, then $d|27$.

Part B

> Show that for all integers n, if n is a multiple of 27, then $n+5$ and n^2+2 are coprime.

2 expressions (for example a and b) are co-prime if $gcd(a,b) = 1$.

By the division algorithm, if $a = bq + r$, then $gcd(a,b) = gcd(b,r)$.

Consider that n^2+2 can be expressed in the form $n^2 - 25 + 27$, which in turn can be factorised as $(n+5)(n-5) + 27$ through application of the "difference of two squares" identity.

If n is a multiple of 27, $n = 27k$, where $k \in \mathbb{Z}$. As a consequence, $n + 5 = 27k + 5$ (1).

Using the division algorithm, $gcd(n^2 + 2, n + 5) = gcd(n + 5, 27)$ (2).

Given (1), equation (2) can be simplified to $gcd(5, 27)$. 5 is a prime number, and not a factor of 27, meaning the GCD must be 1.

Since the greatest common divisor of $n^2 + 2$ and $n + 5$ is 1, the two expressions are co-prime.