Question 1

> Prove that $\sqrt[3]{361}$ is irrational.

To determine whether the cube root of 361 is irrational, we need to examine whether it can be expressed as a rational number, which is a number that can be written as the quotient of two integers (where the denominator is not zero).
Let us assume that $\sqrt[3]{361}$ is a rational number. As such, $\sqrt[3]{361} = \frac{p}{q}$, where p,q are co-prime integers, and $q \neq 0$ (definition of a rational number).

Taking the cube of both sides of this equation gives

$$\begin{aligned} 361 &= \left(\frac{p}{q}\right)^3 \\ &= \frac{p^3}{q^3} \\ p^3 &= 361q^3. \qquad (1) \end{aligned}$$

Since p and q are integers, p^3 and q^3 are also integers.

Consider the prime factorisation $361 = 19 \times 19$. 19 is prime, appears twice in the factorisation, and also is the only number in the factorisation. It is not in the form $\frac{p^3}{q^3}$. Therefore, there is no way to express 361 as the cube of a rational number.

This contradicts our original assumption that $\sqrt[3]{361}$ is rational.

Hence, by contradiction, we can conclude that $\sqrt[3]{361}$ is irrational.

Question 3

Part 1

> Show that for all integers n, if d is an integer such that $d|n+5$ and $d|n^2+2$, then $d|27$.

Let d be an integer. The three requirements for d are that

$$\begin{aligned} d|n+5 \quad & (1) \\ d|n^2+2 \quad & (2) \\ d|27 \quad & (3), \end{aligned}$$

Where n is any integer.

Assume that (1) and (2) are true. The consequence of this is that there must be 2 integers, (let them be a and b), such that

$$\begin{aligned} n+5 &= ad \quad (3), \text{derived from } (1) \\ n^2+2 &= bd \quad (4), \text{derived from } (2). \end{aligned}$$

Equation (3) can be rearranged such that $n = ad - 5$. Substituting this into (4),

$$\begin{aligned} bd &= (ad+5)^2 + 2 \\ &= a^2d^2 - 10ad + 25 + 2 \\ 27 &= bd - a^2d^2 + 10ad \\ 27 &= d\left(b - a^2d + 10a\right). \qquad (5) \end{aligned}$$

Since a,b, and d are all defined as integers, the right hand side of (5) is an integer which is divisible by d. Therefore, 27 must also be divisible by d, satisfying (3). As such, we can conclude that for all integers n, if d is an integer such that $d|n+5$ and $d|n^2+2$, then $d|27$.

Part 2

> Show that for all integers n, if n is a multiple of 27, then $n+5$ and n^2+2 are coprime.

2 expressions (for example a and b) are co-prime if $gcd(a,b)=1$.

By the division algorithm, if $a=bq+r$, then $gcd(a,b)=gcd(b,r)$.

Consider that n^2+2 can be expressed in the form $n^2-25+27$, which in turn can be factorised as $(n+5)(n-5)+27$ through application of the "difference of two squares" identity.

If n is a multiple of 27, $n=27k$, where $k \in \mathbb{Z}$. As a consequence, $n+5=27k+5$ (1).

Using the division algorithm, $gcd(n^2+2, n+5)=gcd(n+5, 27)$ (2).

Given (1), equation (2) can be simplified to $gcd(5,27)$. 5 is a prime number, and not a factor of 27, meaning the GCD must be 1.

Since the greatest common divisor of n^2+2 and $n+5$ is 1, the two expressions are co-prime.